A real life Phenomenon that we study:
A Population Growth in a region.

We considered two models:

$\boxed{\text{Model 1}}$ The Exponential Growth Model.

Assume that there is no immigration or emigra then the rate at which the population is changing is proportional to the population. In otherwords, the larger the population, the faster it is growing. Then if the population at time t is P, then the model is written as follows:

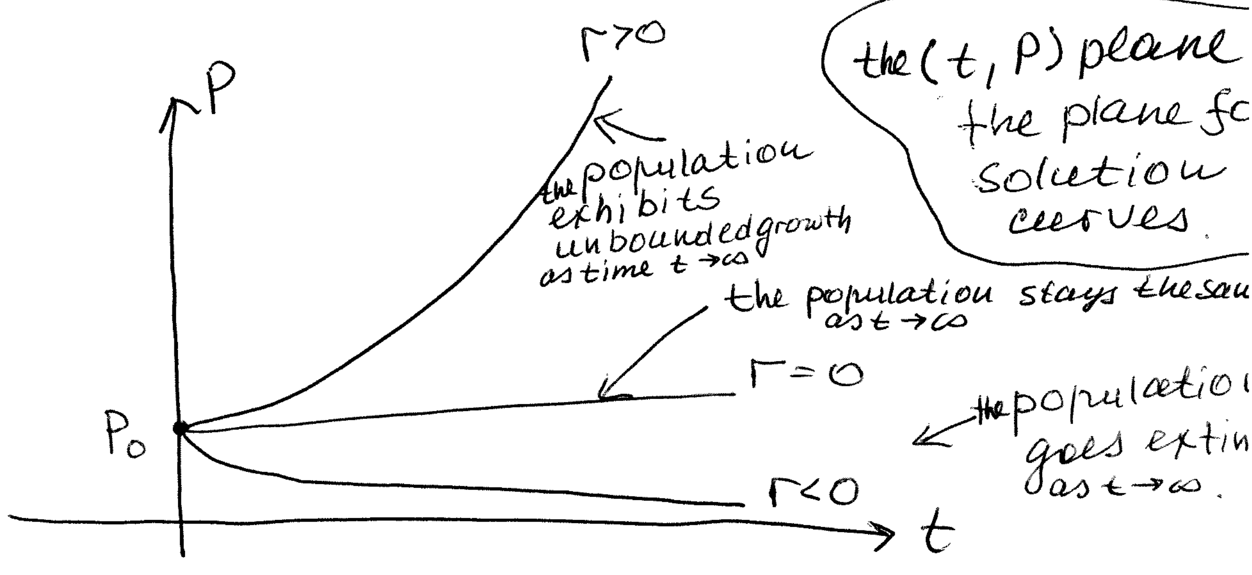
$$\boxed{\begin{aligned} \frac{dP}{dt} &= rP \\ P(0) &= P_0 \end{aligned}}$$

, r = constant, per capita growth rate.

← the initial size of the population.

The solution to Model 1 can be written in the form $P(t) = P_0 \cdot e^{rt}$ to the model.

$\left[\begin{array}{l} P = 0 \text{ is also the solution} \\ P_0 = 0 \end{array}\right.$



Model 2 The Logistic Equation

The equation describes the change in size of a population for which per capita growth depends on population size

$$\begin{cases} \dfrac{dP}{dt} = h(P)\cdot P = \underbrace{r\left(1-\dfrac{P}{K}\right)}_{h(P)}\cdot P \\ P(0) = P_0 \end{cases}$$

The quantity K is called the carrying capacity.

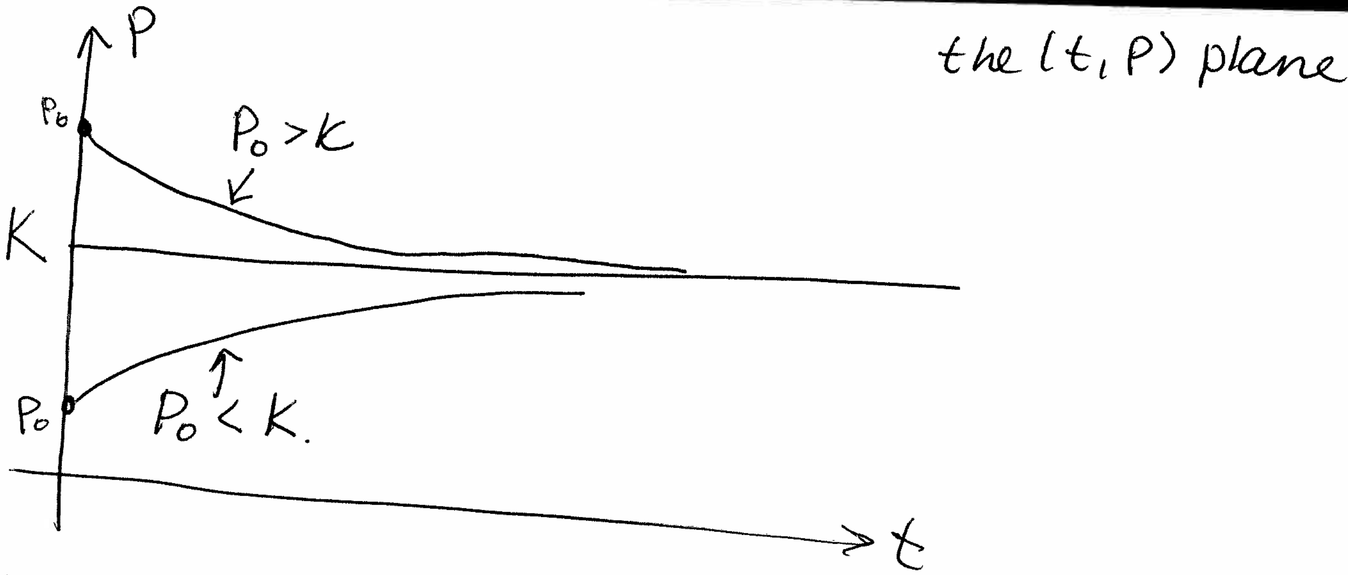
→ If $P > K$ $\quad \dfrac{P}{K} > 1$ and $\underbrace{\left(1-\dfrac{P}{K}\right)}_{h(P)} < 0$

$\Rightarrow \dfrac{dP}{dt} < 0$ when $P > K$. (the population decreases in size), as $t \to \infty$

→ If $P < K$, then $\dfrac{P}{K} < 1$ and $h(P) = \left(1-\dfrac{P}{K}\right) > 0$ and $\dfrac{dP}{dt} > 0$ then the size of the population increases as $t \to \infty$.

→ If $P \equiv K$ then $\dfrac{dP}{dt} = 0$ and the population stays at K.

$$P(t) = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right)e^{-rt}} \longrightarrow K \text{ as } t \to \infty.$$



$P \equiv 0$ is another solution that satisfies Model 2.
($P_0 = 0 \Rightarrow P(t) = 0$ as $t \to \infty$.)
If there aren't any individuals to begin with, there won't be any later on.

$P_0 \equiv K \Rightarrow P(t) = K$ as $t \to \infty$ is another constant solution.

If $P_0 < K$ the population growth is almost like the exponential growth (model 1)

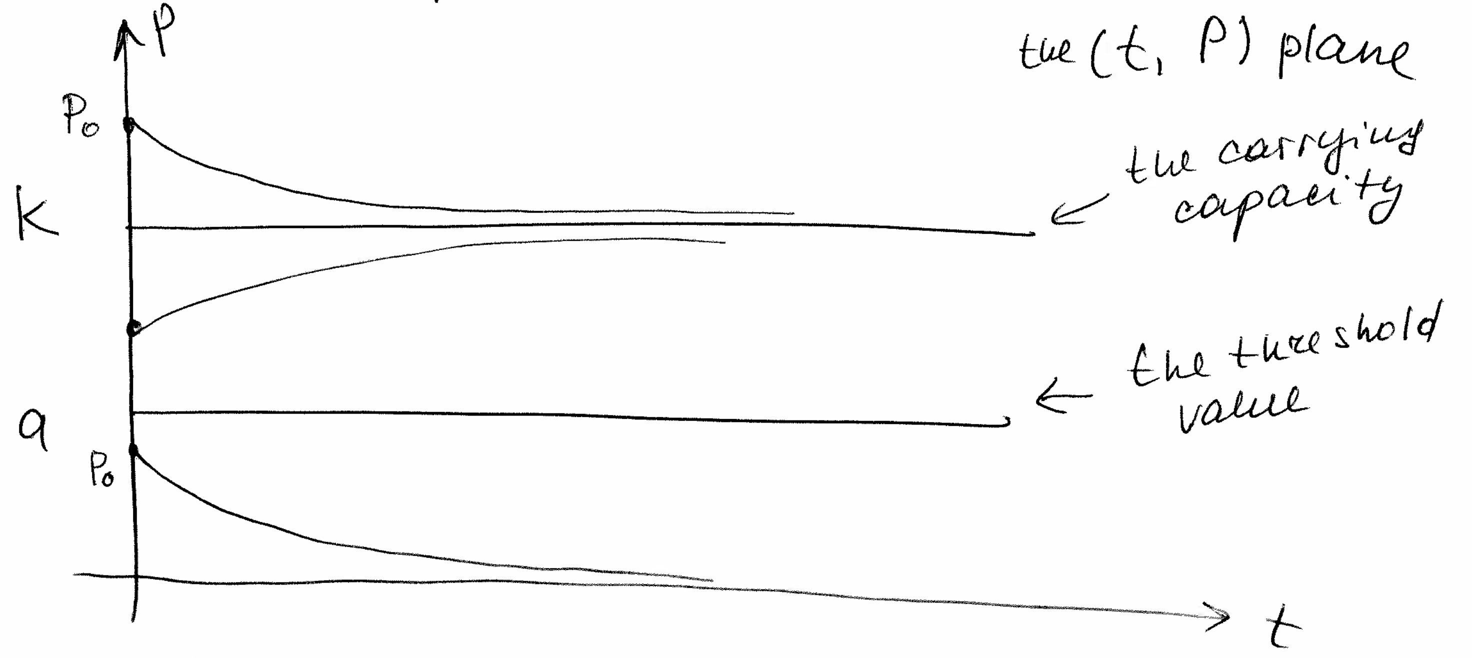
If $P_0 > K$, the population size decreases and becomes asymptotically equal to the carrying capacity

$\boxed{\text{Model 3}}$ Modified Logistic Growth or (The Allee Effect Model)

The previous model predists that any popul
no matter how small, will grow and sooner or later will reach the carrying capacity K.
In the general case, this is not true: if a population falls to a certain minimum, it will keep decreasing and eventually becomes extin
This phenomenon is known as Allee Effect (is -american zoologist.
To build a model that accounts for this fac we adjust Model 3:

$$\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right)\cdot\underbrace{(P-a)}_{\uparrow\text{the new term.}}$$

a, K, r are positive constants.



If $P_0 < a \Rightarrow P(t) \to 0$ as $t \to \infty$. If $P_0 > a$ $P(t) \to K$, $t \to \infty$.

$\boxed{\text{Model 4}}$ Newton's Law of Heating and Coo

Newton proposed that the temperature of a hot object decreases at a rate proportional to the difference between its temperature and that of its surroundings.

Similarly, a cold object heats up at a rate proportional to the temperature difference between the object and its surroundings

Denote the temperature of an object by H and the temperature of the surrounding air by A. Then if $\boxed{H > A}$ then

$$\frac{dH}{dt} = \alpha\,(\underbrace{H-A}_{>0}) < 0,$$

$$\frac{dH}{dt} < 0 \implies \text{constant } \alpha < 0$$

or

$$\frac{dH}{dt} = \alpha(\underbrace{A-H}_{<0}) < 0 \implies \alpha > 0.$$

Cooling

$$\frac{dH}{dt} = \alpha(H-A), \quad H > A, \quad \alpha < 0$$

$$\frac{dH}{H-A} = \alpha\, dt$$

$$\int \frac{dH}{H-A} = \int \alpha\, dt$$

$$\ln|H-A| = \alpha t + C$$

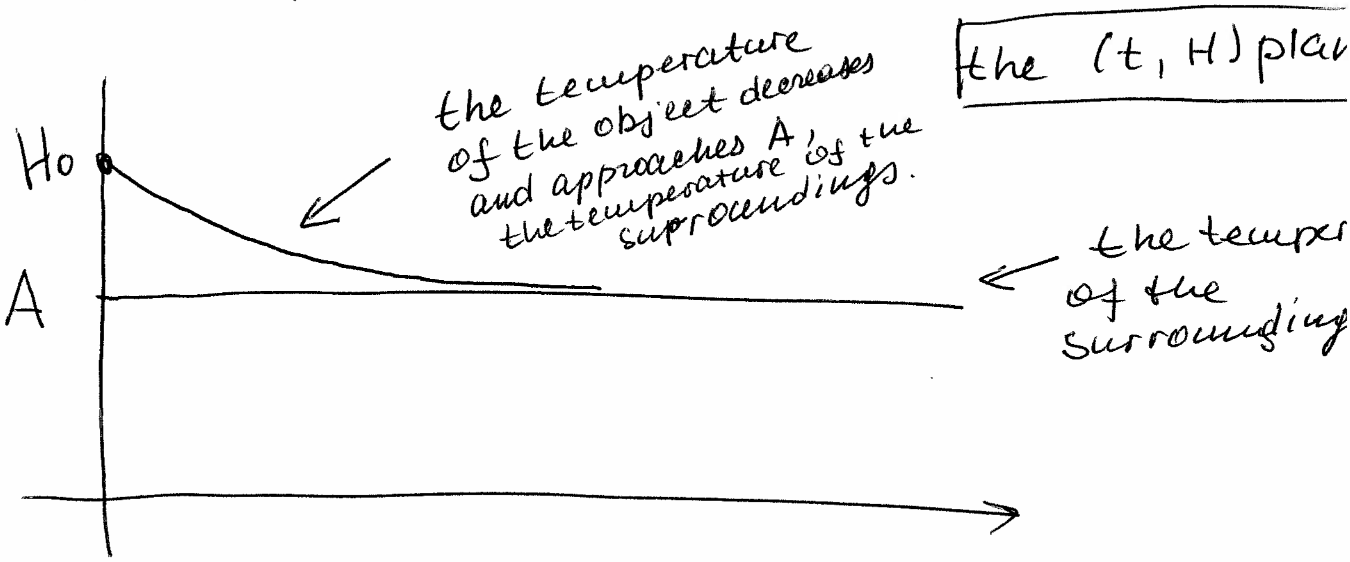
$$|H-A| = e^{\alpha t + C} = e^{\alpha t} \cdot \underbrace{e^{C}}_{= \bar{C}} = \bar{C} \cdot e^{\alpha t}$$

multiplicat

$$H - A = \pm \bar{C} \cdot e^{\alpha t} = \hat{C} \cdot e^{\alpha t}$$

$$H(t) = A + \hat{C} \cdot e^{\alpha t}, \text{ where } \boxed{\alpha < 0}$$

the (t, H) plan



$$H(t) = A + \hat{C} e^{\alpha t} \to A \text{ as } t \to \infty$$

since α is negative constant.

($e^{\alpha t} \to 0$ as $t \to \infty$, because α is negative;

for example $\alpha = -0.1$ then $e^{-0.1t} = \frac{1}{e^{0.1t}} \to$ c

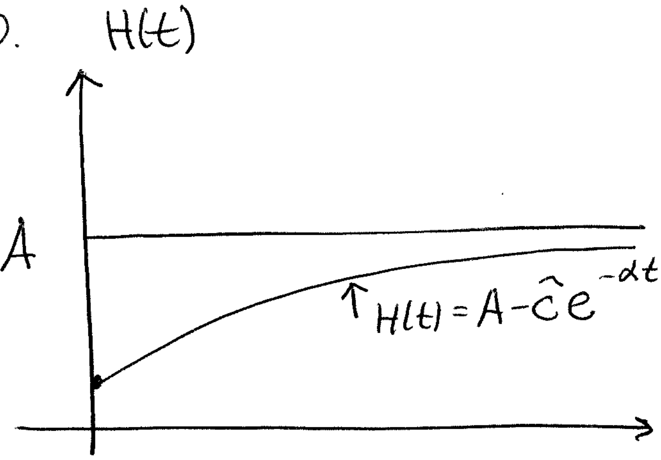
as $t \to \infty$.)

Similarly, a cold object heats up at a rate proportional to the temperature difference between the object and its surroundings (A).

Namely, if $\boxed{H < A}$ then

$$\left.\begin{array}{l} \dfrac{dH}{dt} = \alpha (\underset{\hat{0}}{H-A}) > 0 \quad \Rightarrow \alpha < 0 \\[2ex] \boxed{\text{or}} \\[2ex] \dfrac{dH}{dt} = \alpha (\underset{\check{0}}{A-H}) > 0 \quad \Rightarrow \alpha > 0 \end{array}\right] \text{Heating.}$$

$\dfrac{dH}{dt} = \alpha(A-H) > 0, \quad H < A, \quad \alpha > 0.$



$$\frac{dH}{A-H} = \alpha dt$$

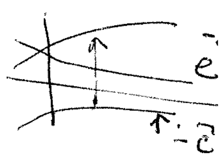
$$\int \frac{dH}{A-H} = \int \alpha dt$$

$$- \ln |A-H| = \alpha t + C$$

do not forget it. if A−H, not H−A.

$$\ln |A-H| = -\alpha t - C$$

$$|A-H| = e^{-\alpha t} \cdot e^{-C} = \bar{C} \cdot e^{-\alpha t}$$



$$A - H = \pm \bar{C} \cdot e^{-\alpha t} = \hat{C} \cdot e^{-\alpha t}$$

$$H(t) = A - \hat{C} e^{-\alpha t}, \quad \alpha > 0.$$

$$H(t) \rightarrow A \text{ as } t \rightarrow \infty.$$

$\boxed{\text{Example}}$ We have a big pot of soup. The soup had just boiled at 100 °C. Our fridge is not powerful enough to accomodate a big pot of soup if it is any warmer than 20 °C. If we put the pot in a sink full of cold water (keep it running, so that its temperature ≈ constant at 5°C) and stirring ocasionally, then we could bring the temperature of the soup to 60° in 10 minutes. How long would it take for the pot to reach 20°C?

$H(0) = 100\,°C$

$A = 5\,°C.$

Newton's Law of Cooling says that for some constant α:

Rate of change of temperature $= \alpha \cdot$ Temperature difference

$$\frac{dH}{dt} = \alpha \cdot (H - A)$$

In our case, $\dfrac{dH}{dt} = \alpha(\underbrace{H-5}_{>0}) < 0$, $\boxed{\alpha < 0}$

H is falling, so the rate of change of the temperature must be negative, α must be negative!

$$\frac{dH}{H-5} = \alpha dt$$

$$\int \frac{dH}{H-5} = \int \alpha dt$$

$$\ln|H-5| = \alpha t + C.$$

$$|H-5| = e^{\alpha t + C} = e^{\alpha t} \cdot e^{C} = \bar{C} \cdot e^{\alpha t}$$

$$H-5 = \pm \bar{C} e^{\alpha t} = \hat{C} e^{\alpha t}$$

$$H(t) = 5 + \hat{C} \cdot e^{\alpha t}$$

α - ? $\hat{C}$ - ? - are unknown constants.

We need to use a given information to find the constants. Namely, we know that we could bring the temperature of the soup to 60°C in 10 minutes. and initially, the pot was at 100°C

$$\begin{cases} 60 = 5 + \hat{C} e^{\alpha \cdot 10} \\ 100 = 5 + \hat{C} e^{\alpha \cdot 0} \end{cases}$$

$$55 = \hat{C} e^{\alpha \cdot 10}$$

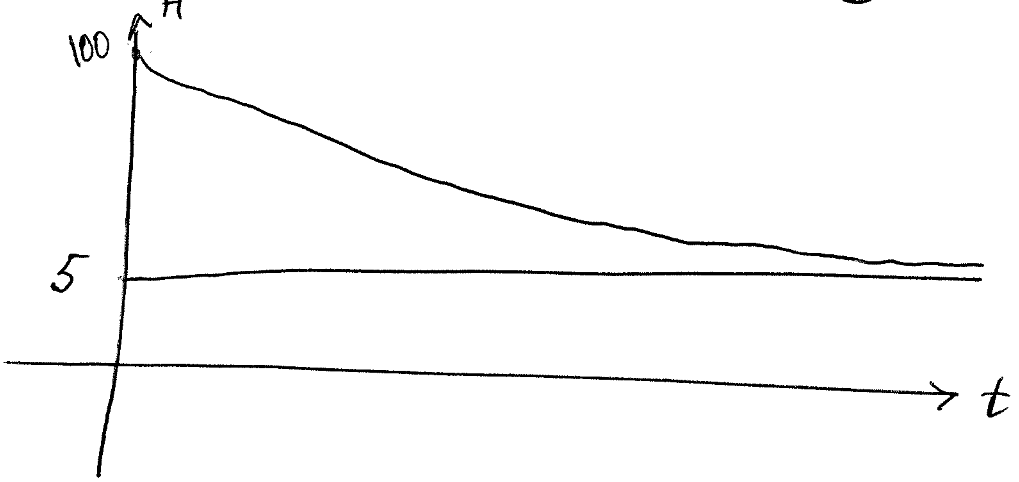
$$95 = \hat{C} \Rightarrow 55 = 95 e^{10\alpha}$$

$$\Rightarrow \frac{55}{95} = e^{10\alpha} \Rightarrow \ln\left(\frac{55}{95}\right) = \ln(e^{10\alpha})$$

$$10\alpha = \ln\left(\frac{55}{95}\right)$$

$\alpha = \frac{1}{10} \ln\left(\frac{55}{95}\right) = -0.054 < 0$

$H(t) = 5 + \hat{C} e^{\alpha t} = 5 + 95 \cdot e^{-0.054t}$



the (t, H) plane.

? ~~How long should the soup be ready~~
so we could put it in the fridge.

We Obtained, $\boxed{H(t) = 5 + 95 e^{-0.054t}}$

We can put the pot in the fridge only if the temperature of the pot is less or equal to 20°
When does it happen?

$20 = 5 + 95 e^{-0.054T}$

Find T.

$15 = 95 e^{-0.054T}$

$\frac{15}{95} = e^{-0.054T}$

$\ln\left(\frac{15}{95}\right) = \ln\left(e^{-0.054T}\right) = -0.054T.$

$$T = \frac{\ln\left(\frac{15}{95}\right)}{-0.054} \approx 34 \text{(minutes)}$$

It will take a little over half an hour for soup to cool off enough to be put into the fridge.

If the temperature of the soup at 4 p.m. is 40°C, when was the soup cooked?

$$H(t) = 5 + 95e^{-0.054t}$$ ← the soup cools off according to this eq-n.

$$40 = 5 + 95e^{-0.054T} \qquad T - ?$$

$$\frac{35}{95} = e^{-0.054T}$$

$$T = -\frac{1}{0.054}\ln\left(\frac{35}{95}\right) \approx 18 \text{ (min)}.$$

$16{:}00 - T$ = time when the was cooked.

The soup was cooked at 3:42 p.m (or 15:42)